Exact representation of the unit step function through algebraic functions

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Abstract

In this paper, the author obtains an algebraic representation of the unit step function, known as Heaviside function, using, a combination of three algebraic functions: linear, quadratic and irrational. Heaviside step function is widely used in operational calculus, control theory, signal processing theory and describing transients. The use of elementary algebraic functions for accurate representation of step functions will simplify mathematical models of piecewise continuous processes and computational procedures in many ways. An example of describing transients process in a special dynamic system using injected presentation of the unit step function is considered here.

Keywords

The unit step function, Heaviside step function, the algebraic representation, describing transients.

1. Introdaction

Heaviside step function , or the unit step function, is a piecewise constant function which equals zero for negative values of the argument, and unit – for positive ones. There are two definitions of this function [1]

$$H(x)=\begin{cases}0, & x<0;\\ \frac{1}{2}, & x=0;\\ 1, & x>0.\end{cases} \quad \text{и} \quad H(x)=\begin{cases}0, & x<0;\\ 1, & x=0.\end{cases} \qquad (1)$$

Heaviside step functions can be defined by the following different analytical forms [1-5]

$$H(x)\approx\frac{1}{2}+\frac{1}{2}\operatorname{th}kx=\frac{1}{1+e^{-2kx}},$$

$$H(x)=\lim_{k\to\infty}\frac{1}{2}(1+\operatorname{th}kx)=\lim_{k\to\infty}\frac{1}{1+e^{-2kx}},$$

$$H(x)=\lim_{k\to\infty}\left(\frac{1}{2}+\frac{1}{\pi}a\tan kx\right),$$

$$H(x)=\lim_{k\to\infty}\frac{1}{2}(1+\operatorname{erf} kx),$$

$$H(x)=-\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_0^{\infty}\frac{1}{\tau+i\varepsilon}e^{-ix\tau}\,d\tau .$$

Heaviside step function also can be defined in terms of the sign function by

$$H(x)=\frac{1}{2}(1+\operatorname{sgn} x). \qquad (2)$$

Sullivan et al [6] obtained a linear algebraic approximation of this function by means of a linear combination of exponential functions. It has been shown that the functions which can be used for approximating unit step functions have the form

$$F_N(t)=\sum_{n=1}^{N}A_n^{(N)}\,e^{-\lambda_n t/T},$$

N – is a given integer ≥ 1; λ_n – are fixed numbers: $0<\lambda_1<\lambda_2<\ldots<\lambda_N$; The coefficients $A_n^{(N)}$ are found by requiring that they minimize the integral

$$\int_0^{\infty}[f(t)-F_N(t)]^2\,dt,$$

where the unit step function $f(t)$ is defined in the form:

$$f(t)=1, \quad 0\leq t\leq T\ (T>0);$$

$$f(t)=1, \quad t>T.$$

The analytically exact shape of the unit step function in closed form as a sum of two inverse trigonometric functions is obtained by J.Venetis [7]

$$H(x)=\frac{1}{\pi}\left(\frac{3\pi}{4}+\arctan(x-1)+\arctan\left(\frac{2-x}{x}\right)\right).$$

2. The algebraic representation of the unit step function

Based on the relation of the Heaviside function to the sign function (2), the representation of the unit step function can be performed in the following forms

$$H(x)=\frac{1}{2}\left(1+\frac{x}{|x|}\right) \tag{3}$$

and

$$H(x)=\frac{1}{2}\left(1+\frac{x}{\sqrt{x^2}}\right). \tag{4}$$

Representation (3) and (4) are equivalent. They define the unit step function which is not defined at the point $x=0$, but can be clearly defined at this point to meet one of the definitions (1) the Heaviside function.

The unit function "step-up" at the point c is defined by the equality

$$H_c(x)=\frac{1}{2}\left(1+\frac{x-c}{\sqrt{(x-c)^2}}\right).$$

The unit function "step-down" at the point c is defined by the equality

$$\chi_c(x)=1-H_c(x)=\frac{1}{2}\left(1+\frac{c-x}{\sqrt{(x-c)^2}}\right).$$

The unit function with a "step-up" at the point a and a "step-down" at the point b $(b>a)$ is defined by the equality

$$H_{ab}(x)=H_a(x)+H_b(x)=1+\frac{1}{2}\left(\frac{x-a}{\sqrt{(x-a)^2}}+\frac{x-b}{\sqrt{(x-b)^2}}\right).$$

The equivalent forms of recording of function $H_{ab}(x)$ are possible

$$H_{ab}(x)=1+H_a(x)-\chi_b(x),$$

as well as

$$H_{ab}(x)=H_a(x)\chi_b(x),$$

$$H_{ab}(x)=\frac{1}{2}\left(1+\frac{(x-a)}{\sqrt{(x-a)^2}}\cdot\frac{(b-x)}{\sqrt{(x-b)^2}}\right).$$

3. Example of the transition process in a special dynamic system

As an example, it is considered the continuous description of the transition process in a dynamic system given by the equations

$$\begin{cases} \dfrac{dx}{dt} = v, \\ \dfrac{dv}{dt} = f(t), \end{cases} \quad 0 \le t \le \tau; \qquad \begin{cases} \dfrac{dx}{dt} = u, \\ \dfrac{du}{dt} = -k^2 x - 2nu, \end{cases} \quad n > 0; \;\; \tau \le t < \infty;$$

The function $f(t)$ is selected from the conditions of the stationary initial state of the system and smooth pasting phase trajectories

$$v(0) = 0, \; f(0) = 0, \; v(\tau) = u(\tau), \; f(\tau) = -kx(\tau) - bu(\tau).$$

as a polynomial of the fifth degree

$$f(t) = \sum_{i=1}^{5} a_i t^i, \quad 0 \le t \le \tau.$$

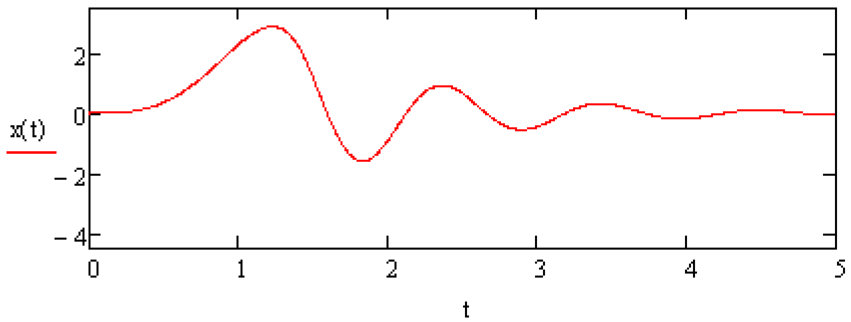
This allows us to record continues process of the stabilization and perturbation of a dynamical system by one equation. At $n < k$

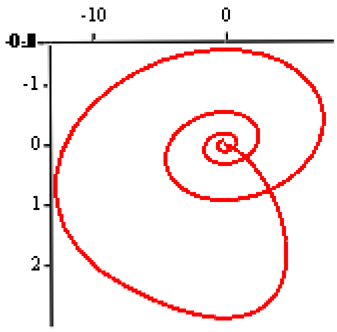
$$x = H_{0\tau} \sum_{i=1}^{5} a_i t^i + (1 - H_{0\tau}) A e^{-nt} \sin\left(\sqrt{k^2 - n^2}\, t + \alpha\right), \quad 0 \le t < \infty$$

and

$$x = \left(\frac{1}{2}\left(1 + \frac{t}{\sqrt{t^2}} \cdot \frac{(\tau - t)}{\sqrt{(t-\tau)^2}}\right)\right) \sum_{i=1}^{5} a_i t^i + \left(\frac{1}{2}\left(1 - \frac{t}{\sqrt{t^2}} \cdot \frac{(\tau - t)}{\sqrt{(t-\tau)^2}}\right)\right) A e^{-nt} \sin\left(\sqrt{k^2 - n^2}\, t + \alpha\right), \quad 0 \le t < \infty$$

Figure 1 shows the transition process and its phase trajectory for the parameter values: $\tau = 1.4, \quad k = 6, \quad n = 1, \quad A = 10, \quad \alpha = 0$





. Figure 1. A plot of the transition process and the phase trajectory

4. Discussion

In addition to the results [7] another option of an algebraic representation of the unit step function is proposed. The representation given here doesn't have the disadvantage mentioned in work [7] that the inverse trigonometric functions don't have a unique definitions Simplicity of the analytical recording of the unit step function can be successfully used in a variety of engineering applications in mathematical modeling of systems and processes which use piecewise discontinuous, piecewise continuous and piecewise smooth functions.

References

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